A geometric realization of spin representations and Young diagrams from quiver varieties

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Abstract

Applying the techniques of an earlier paper with Frenkel, we develop a geometric realization of spin representations and Clifford algebras. In doing so, we give an explicit parametrization of the irreducible components of Nakajima varieties of type D in terms of Young diagrams. We explicity compute the geometric action of the Lie algebra and are able to extend the geometric action to the entire Clifford algebra used in the classical construction of the spin representations.

Introduction

In [FS03], we related two apparently different bases in the representations of affine Lie algebras of type A: one arising from statistical mechanics, the other from gauge theory. In particular, using geometric methods associated to quiver varieties, we were able to give an alternative and much simpler geometric proof of a result of [DJKMO89] on the construction of bases of affine Lie algebra representations. At the same time, we gave a simple parametrization of the irreducible components of Nakajima quiver varieties associated to infinite and cyclic quivers in terms of Young and Maya diagrams. In the current paper, we consider the spin representations of the Lie algebra $\mathfrak{g} = \mathfrak{so}_{2n}\mathbb{C}$ of type D_n. Applying the techniques of [FS03], we are able to give a very explicit parametrization of the irreducible components of the associated Nakajima varieties in terms of Young diagrams of strictly decreasing row length with maximum length $n-1$. We also explicitly compute the geometric action and thus obtain a realization of the spin representations in terms of these diagrams. Furthermore, we are able to extend the geometric action to the entire Clifford algebra used in the classical construction of the spin representations. This is the first example of a geometric realization of additional structure on irreducible representations extending the module structure originally defined by Nakajima. Possible extensions of these results to the affine case may suggest new statistical mechanics results for type D analogous to those of type A mentioned above.

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1 Lusztig's Quiver Varieties

In this section, we will recount the explicit description given in [L91] of the irreducible components of Lusztig's quiver variety for type D_n. See this reference for details, including proofs.

1.1 General Definitions

Let I be the set of vertices of the Dynkin graph of a symmetric Kac-Moody Lie algebra $\mathfrak{g}$ and let H be the set of pairs consisting of an edge together with an orientation of it. For $h \in H$, let $\mathrm{in}(h)$ (resp. $\mathrm{out}(h)$) be the incoming (resp. outgoing) vertex of h. We define the involution $\bar{\ } : H \to H$ to be the function which takes $h \in H$ to the element of H consisting of the same edge with opposite orientation. An *orientation* of our graph is a choice of a subset $\Omega \subset H$ such that $\Omega \cup \bar{\Omega} = H$ and $\Omega \cap \bar{\Omega} = \emptyset$.

Let $\mathcal{V}$ be the category of finite-dimensional I-graded vector spaces $\mathbf{V} = \oplus_{i \in I} \mathbf{V}_i$ over $\mathbb{C}$ with morphisms being linear maps respecting the grading. Then $\mathbf{V} \in \mathcal{V}$ shall denote that $\mathbf{V}$ is an object of $\mathcal{V}$. Labelling the vertices of I by $1, \dots, n$, the dimension of $\mathbf{V} \in \mathcal{V}$ is given by $\mathbf{v} = \dim \mathbf{V} = (\dim \mathbf{V}_1, \dots, \dim \mathbf{V}_n)$.

Given $\mathbf{V} \in \mathcal{V}$, let

$$\mathbf{E}_{\mathbf{V}} = \bigoplus_{h \in H} \mathrm{Hom}(\mathbf{V}_{\mathrm{out}(h)}, \mathbf{V}_{\mathrm{in}(h)}).$$

For any subset H' of H, let $\mathbf{E}_{\mathbf{V},H'}$ be the subspace of $\mathbf{E}_{\mathbf{V}}$ consisting of all vectors $x = (x_h)$ such that $x_h = 0$ whenever $h \notin H'$. The algebraic group $G_{\mathbf{V}} = \prod_i \mathrm{Aut}(\mathbf{V}_i)$ acts on $\mathbf{E}_{\mathbf{V}}$ and $\mathbf{E}_{\mathbf{V},H'}$ by

$$(g, x) = ((g_i), (x_h)) \mapsto gx = (x'_h) = (g_{\mathrm{in}(h)} x_h g_{\mathrm{out}(h)}^{-1}).$$

Define the function $\varepsilon : H \to \{-1, 1\}$ by $\varepsilon(h) = 1$ for all $h \in \Omega$ and $\varepsilon(h) = -1$ for all $h \in \bar{\Omega}$. For $\mathbf{V} \in \mathcal{V}$, the Lie algebra of $G_{\mathbf{V}}$ is $\mathbf{gl}_{\mathbf{V}} = \prod_i \mathrm{End}(\mathbf{V}_i)$ and it acts on $\mathbf{E}_{\mathbf{V}}$ by

$$(a, x) = ((a_i), (x_h)) \mapsto [a, x] = (x'_h) = (a_{\mathrm{in}(h)} x_h - x_h a_{\mathrm{out}(h)}).$$

Let $\langle \cdot, \cdot \rangle$ be the nondegenerate, $G_{\mathbf{V}}$-invariant, symplectic form on $\mathbf{E}_{\mathbf{V}}$ with values in $\mathbb{C}$ defined by

$$\langle x, y \rangle = \sum_{h \in H} \varepsilon(h) \,\mathrm{tr}(x_h y_{\bar{h}}).$$

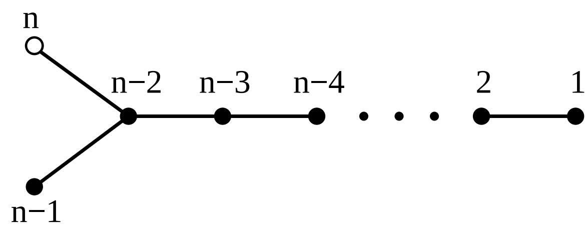

Figure 1: The Dynkin graph of type D_n. We represent the nth vertex by an open dot to distinguish it from the $(n-1)$st vertex.

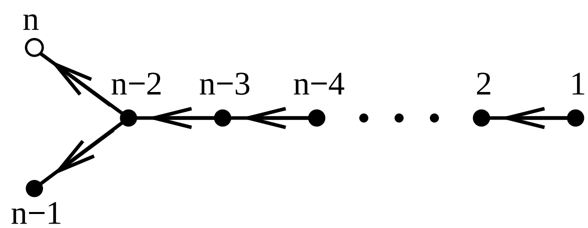

Figure 2: The quiver (oriented graph) of type D_n.

Note that $\mathbf{E_V}$ can be considered as the cotangent space of $\mathbf{E}_{\mathbf{V},\Omega}$ under this form.

The moment map associated to the $G_{\mathbf{V}}$-action on the symplectic vector space $\mathbf{E_V}$ is the map $\psi : \mathbf{E_V} \to \mathbf{gl_V}$ with i-component $\psi_i : \mathbf{E_V} \to \operatorname{End} \mathbf{V}_i$ given by

$$\psi_i(x) = \sum_{h \in H,\, \mathrm{in}(h)=i} \varepsilon(h) x_h x_{\bar{h}}.$$

Definition 1.1 ([L91]). *An element $x \in \mathbf{E_V}$ is said to be* nilpotent *if there exists an $N \geq 1$ such that for any sequence $h'_1, h'_2, \dots, h'_N$ in H satisfying $\mathrm{out}(h'_1) = \mathrm{in}(h'_2)$, $\mathrm{out}(h'_2) = \mathrm{in}(h'_3)$, $\dots$, $\mathrm{out}(h'_{N-1}) = \mathrm{in}(h'_N)$, the composition $x_{h'_1} x_{h'_2} \dots x_{h'_N} : \mathbf{V}_{\mathrm{out}(h'_N)} \to \mathbf{V}_{\mathrm{in}(h'_1)}$ is zero.*

Definition 1.2 ([L91]). *Let $\Lambda_{\mathbf{V}}$ be the set of all nilpotent elements $x \in \mathbf{E_V}$ such that $\psi_i(x) = 0$ for all $i \in I$.*

1.2 Type D_n

Let $\mathfrak{g} = \mathfrak{so}_{2n}\mathbb{C}$ be the simple Lie algebra of type D_n. Let $I = \{1, 2, \dots, n\}$ be the set of vertices of the Dynkin graph of $\mathfrak{g}$, labelled as in Figure 1. We let Ω be the orientation indicated in Figure 2.

Label each oriented edge by its incoming and outgoing vertices. That is, if vertices i and j are connected by an edge, $h_{i,j}$ denotes the oriented edge with $\mathrm{out}(h) = i$, $\mathrm{in}(h) = j$. The following is proven in [L91, Proposition 14.2].

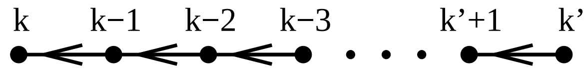

Figure 3: The string representing $(\mathbf{V}(k',k), x(k',k))$, $1 \le k' \le k \le n-1$.

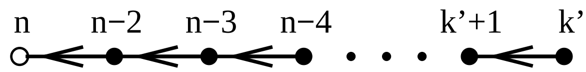

Figure 4: The string representing $(\mathbf{V}(k',n), x(k',n))$, $1 \le k' \le n-2$ or $k' = n$.

Proposition 1.3. *The irreducible components of $\Lambda_{\mathbf{V}}$ are the closures of the conormal bundles of the various $G_{\mathbf{V}}$-orbits in $\mathbf{E}_{\mathbf{V},\Omega}$.*

For two integers $1 \le k' \le k \le n-1$, define $\mathbf{V}(k',k) \in \mathcal{V}$ to be the vector space with basis $\{e_r \mid k' \le r \le k\}$. We require that e_r has degree $r \in I$. Let $x(k',k) \in \mathbf{E}_{\mathbf{V}(k',k),\Omega}$ be defined by $x(k',k)_{h_{r,r+1}} : e_r \mapsto e_{r+1}$ for $k' \le r < k$, and all other components of $x(k',k)$ are zero. We picture this representation as the string of Figure 3.

For an integer k' such that $1 \le k' \le n-2$ or $k' = n$, define $\mathbf{V}(k',n)$ to be the vector space with basis $\{e_r \mid k' \le r \le n-2 \text{ or } r = n\}$. Again, we require that e_r has degree $r \in I$. Let $x(k',n) \in \mathbf{E}_{\mathbf{V}(k',n),\Omega}$ be defined by

$$x(k',n)_{h_{r,r+1}}(e_r) = \begin{cases} e_{r+1} & \text{if } r < n-2 \\ 0 & \text{otherwise} \end{cases},$$
$$x(k',n)_{h_{n-2,n}}(e_{n-2}) = e_n$$

and all other components of $x(k',k)$ are zero. We picture this representation as the string of Figure 4.

Next, for an integer k' such that $1 \le k' \le n-2$, define $\mathbf{V}(k',n+1)$ to be the vector space with basis $\{e_r \mid k' \le r \le n\}$ where the degree of e_r is $r \in I$. $x(k',n+1)$ is defined by

$$x(k',n+1)_{h_{r,r+1}}(e_r) = \begin{cases} e_{r+1} & \text{if } r \le n-2 \\ 0 & \text{otherwise} \end{cases},$$
$$x(k',n+1)_{h_{n-2,n}}(e_{n-2}) = e_n$$

and all other components of $x(k',n+1)$ are zero. We picture this representation as the (forked) string of Figure 5.

Finally, for $1 \le k' < k \le n-2$, let $\widetilde{\mathbf{V}}(k',k)$ be the vector space with basis $\{e_r \,|\, k \le r \le n\} \cup \{\tilde{e}_r \,|\, k' \le r \le n-2\}$ where the degree of e_r and $\tilde{e}_r$ is $r \in I$.

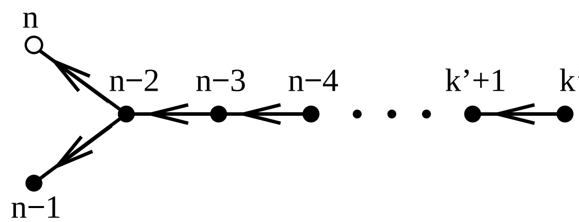

Figure 5: The (forked) string representing $(\mathbf{V}(k', n+1), x(k', n+1))$, $1 \le k' \le n-2$.

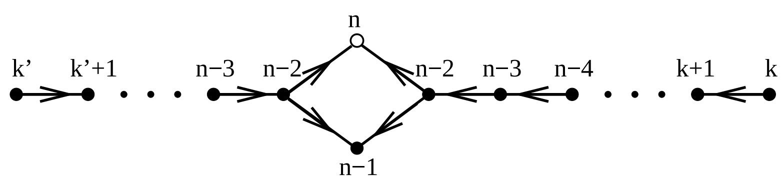

Figure 6: The string representing $(\widetilde{\mathbf{V}}(k', k), \tilde{x}(k', k))$, $1 \le k' < k \le n-2$.

Let $\tilde{x}(k', k)$ be defined by

$$\tilde{x}(k',k)_{h_{r,r+1}}(e_r) = \begin{cases} e_{r+1} & \text{if } r \ge n-2, \\ 0 & \text{otherwise} \end{cases},$$

$$\tilde{x}(k',k)_{h_{r,r+1}}(\tilde{e}_r) = \begin{cases} \tilde{e}_{r+1} & \text{if } r < n-2, \\ e_{n-1} & \text{if } r = n-2 \\ 0 & \text{otherwise} \end{cases},$$

$$\tilde{x}(k',k)_{h_{n-2,n}}(e_{n-2}) = e_n$$
$$\tilde{x}(k',k)_{h_{n-2,n}}(\tilde{e}_{n-2}) = e_n$$

and all other components of $\tilde{x}(k', k)$ are zero. We picture this representation as the string of Figure 6.

Proposition 1.4. *The above* $(\mathbf{V}(k', k), x(k', k))$ *and* $(\widetilde{\mathbf{V}}(k', k), \tilde{x}(k', k))$ *are indecomposable representations of the* D_n *quiver with orientation* Ω. *Conversely, any indecomposable finite-dimensional representation* $(\mathbf{V}, x)$ *of this quiver is isomorphic to one of these representations.*

Proof. It is easy to see that each of the above representations is indecomposable. The fact that our list is exhaustive follows from the fact that the indecomposable representations of the D_n quiver are in one-to-one correspondence with the positive roots of the Lie algebra of type D_n (see [K80, K83]). □

Let

$$\begin{aligned}Z =&\{(k',k)\,|\,(k',k)\neq(n-1,n),(n-1,n+1),(n,n+1),(n+1,n+1)\}\\ &\cup\{(k',k)^{\sim}\,|\,1\le k'<k\le n-2\},\end{aligned}$$

and let $\tilde{Z}$ be the set of all functions $Z\to\mathbb{Z}_{\ge 0}$ with finite support. It is clear that for $\mathbf{V}\in\mathcal{V}$, the set of $G_{\mathbf{V}}$-orbits in $\mathbf{E}_{\mathbf{V},\Omega}$ is naturally indexed by the subset $\tilde{Z}_{\mathbf{V}}$ of $\tilde{Z}$ consisting of those $f\in\tilde{Z}$ such that

$$\sum_{A(k',k)\ni i} f(k',k)+\sum_{B(k',k)\ni i} f((k',k)^{\sim})(\dim\widetilde{\mathbf{V}}(k',k))_i=\dim\mathbf{V}_i\quad\forall\, i\in I$$

where $A(k',k)$ is the set of all $i\in I$ such that $\mathbf{V}(k',k)$ contains a vector of degree i and $B(k',k)$ is the set of all $i\in I$ such that $\widetilde{\mathbf{V}}(k',k)$ contains a vector of degree i. Note that we write $f(k',k)$ for $f((k',k))$. Corresponding to a given f is the orbit consisting of all representations isomorphic to a sum of the indecomposable representations $x(k',k)$ and $\tilde{x}(k',k)$, each occurring with multiplicity $f(k',k)$ and $f((k',k)^{\sim})$ respectively. Denote by $\mathcal{O}_f$ the $G_{\mathbf{V}}$-orbit corresponding to $f\in\tilde{Z}_{\mathbf{V}}$. Let $\mathcal{C}_f$ be the conormal bundle to $\mathcal{O}_f$ and let $\bar{\mathcal{C}}_f$ be its closure. We then have the following proposition.

Proposition 1.5. *The map $f\to\bar{\mathcal{C}}_f$ is a one-to-one correspondence between the set $\tilde{Z}_{\mathbf{V}}$ and the set of irreducible components of $\Lambda_{\mathbf{V}}$.*

Proof. This follows immediately from Propositions 1.3 and 1.4. □

2 Nakajima's Quiver Varieties

We introduce here a description of the quiver varieties first presented in [N94] for type D_n.

Definition 2.1 ([N94]). *For $\mathbf{v},\mathbf{w}\in\mathbb{Z}^I_{\ge 0}$, choose I-graded vector spaces $\mathbf{V}$ and $\mathbf{W}$ of graded dimensions $\mathbf{v}$ and $\mathbf{w}$ respectively. Then define*

$$\Lambda\equiv\Lambda(\mathbf{v},\mathbf{w})=\Lambda_{\mathbf{V}}\times\sum_{i\in I}\mathrm{Hom}(\mathbf{V}_i,\mathbf{W}_i).$$

Now, suppose that $\mathbf{S}$ is an I-graded subspace of $\mathbf{V}$. For $x\in\Lambda_{\mathbf{V}}$ we say that $\mathbf{S}$ is *x-stable* if $x(\mathbf{S})\subset\mathbf{S}$.

Definition 2.2 ([N94]). *Let $\Lambda^{st}=\Lambda(\mathbf{v},\mathbf{w})^{st}$ be the set of all $(x,j)\in\Lambda(\mathbf{v},\mathbf{w})$ satisfying the following condition: If $\mathbf{S}=(\mathbf{S}_i)$ with $\mathbf{S}_i\subset\mathbf{V}_i$ is x-stable and $j_i(\mathbf{S}_i)=0$ for $i\in I$, then $\mathbf{S}_i=0$ for $i\in I$.*

The group $G_{\mathbf{V}}$ acts on $\Lambda(\mathbf{v},\mathbf{w})$ via

$$(g,(x,j))=((g_i),((x_h),(j_i)))\mapsto((g_{\mathrm{in}(h)}x_h g^{-1}_{\mathrm{out}(h)}),(j_i g_i^{-1})).$$

and the stabilizer of any point of $\Lambda(\mathbf{v},\mathbf{w})^{\mathrm{st}}$ in $G_{\mathbf{V}}$ is trivial (see [N98, Lemma 3.10]). We then make the following definition.

Definition 2.3 ([N94]). *Let* $\mathcal{L} \equiv \mathcal{L}(\mathbf{v}, \mathbf{w}) = \Lambda(\mathbf{v}, \mathbf{w})^{st}/G_{\mathbf{V}}$.

Let $\operatorname{Irr}\mathcal{L}(\mathbf{v}, \mathbf{w})$ (resp. $\operatorname{Irr}\Lambda(\mathbf{v}, \mathbf{w})$) be the set of irreducible components of $\mathcal{L}(\mathbf{v}, \mathbf{w})$ (resp. $\Lambda(\mathbf{v}, \mathbf{w})$). Then $\operatorname{Irr}\mathcal{L}(\mathbf{v}, \mathbf{w})$ can be identified with

$$\{Y \in \operatorname{Irr}\Lambda(\mathbf{v}, \mathbf{w}) \,|\, Y \cap \Lambda(\mathbf{v}, \mathbf{w})^{\mathrm{st}} \neq \emptyset\}.$$

Specifically, the irreducible components of $\operatorname{Irr}\mathcal{L}(\mathbf{v}, \mathbf{w})$ are precisely those

$$X_f \stackrel{\text{def}}{=} \left(\left(\bar{\mathcal{C}}_f \times \sum_{i \in I} \operatorname{Hom}(\mathbf{V}_i, \mathbf{W}_i) \right) \cap \Lambda(\mathbf{v}, \mathbf{w})^{\mathrm{st}} \right) / G_{\mathbf{V}}$$

which are nonempty.

The following will be used in the sequel.

Lemma 2.4. *One has*

$$\Lambda^{st} = \left\{ x \in \Lambda \;\middle|\; \ker j_i \cap \bigcap_{h:\mathrm{out}(h)=i} \ker x_h = 0 \;\forall\; i \right\}.$$

Proof. Since each $\bigcap_{h:\mathrm{out}\, h=i} \ker x_h$ is x-stable, the left hand side is obviously contained in the right hand side. Now suppose x is an element of the right hand side. Let $\mathbf{S} = (\mathbf{S}_i)$ with $\mathbf{S}_i \subset \mathbf{V}_i$ be x-stable and $j_i(\mathbf{S}_i) = 0$ for $i \in I$. Assume that $\mathbf{S} \neq 0$. Since all elements of Λ are nilpotent, we can find a minimal value of N such that the condition in Definition 1.1 is satisfied. Then we can find a $v \in \mathbf{S}_i$ for some i and a sequence $h'_1, h'_2, \ldots, h'_{N-1}$ (empty if $N = 1$) in H such that $\mathrm{out}(h'_1) = \mathrm{in}(h'_2)$, $\mathrm{out}(h'_2) = \mathrm{in}(h'_3)$, $\ldots$, $\mathrm{out}(h'_{N-2}) = \mathrm{in}(h'_{N-1})$ and $v' = x_{h'_1} x_{h'_2} \ldots x_{h'_{N-1}}(v) \neq 0$. Now, $v' \in \mathbf{S}_{i'}$ for some $i' \in I$ by the stability of $\mathbf{S}$ (hence $j_{i'}(v') = 0$) and $v' \in \bigcap_{h:\mathrm{out}\, h=i'} \ker x_h$ by our choice of N. This contradicts the fact that x is an element of the right hand side. $\square$

3 The Lie Algebra Action

We summarize here some results from [N94] that will be needed in the sequel. See this reference for more details, including proofs. We keep the notation of Sections 1 and 2.

Let $\mathbf{w}, \mathbf{v}, \mathbf{v}', \mathbf{v}'' \in \mathbb{Z}_{\geq 0}^I$ be such that $\mathbf{v} = \mathbf{v}' + \mathbf{v}''$. Consider the maps

$$\Lambda(\mathbf{v}'', \mathbf{0}) \times \Lambda(\mathbf{v}', \mathbf{w}) \stackrel{p_1}{\leftarrow} \tilde{\mathbf{F}}(\mathbf{v}, \mathbf{w}; \mathbf{v}'') \stackrel{p_2}{\rightarrow} \mathbf{F}(\mathbf{v}, \mathbf{w}; \mathbf{v}'') \stackrel{p_3}{\rightarrow} \Lambda(\mathbf{v}, \mathbf{w}), \tag{3.1}$$

where the notation is as follows. A point of $\mathbf{F}(\mathbf{v}, \mathbf{w}; \mathbf{v}'')$ is a point $(x, j) \in \Lambda(\mathbf{v}, \mathbf{w})$ together with an I-graded, x-stable subspace $\mathbf{S}$ of $\mathbf{V}$ such that $\dim \mathbf{S} = \mathbf{v}' = \mathbf{v} - \mathbf{v}''$. A point of $\tilde{\mathbf{F}}(\mathbf{v}, \mathbf{w}; \mathbf{v}'')$ is a point $(x, j, \mathbf{S})$ of $\mathbf{F}(\mathbf{v}, \mathbf{w}; \mathbf{v}'')$ together with a collection of isomorphisms $R'_i : \mathbf{V}'_i \cong \mathbf{S}_i$ and $R''_i : \mathbf{V}''_i \cong \mathbf{V}_i/\mathbf{S}_i$ for each

$i \in I$. Then we define $p_2(x, j, \mathbf{S}, R', R'') = (x, j, \mathbf{S})$, $p_3(x, j, \mathbf{S}) = (x, j)$ and $p_1(x, j, \mathbf{S}, R', R'') = (x'', x', j')$ where x'', x', j' are determined by

$$\begin{aligned} R'_{\mathrm{in}(h)} x'_h &= x_h R'_{\mathrm{out}(h)} : \mathbf{V}'_{\mathrm{out}(h)} \to \mathbf{S}_{\mathrm{in}(h)}, \\ j'_i &= j_i R'_i : \mathbf{V}'_i \to \mathbf{W}_i \\ R''_{\mathrm{in}(h)} x''_h &= x_h R''_{\mathrm{out}(h)} : \mathbf{V}''_{\mathrm{out}(h)} \to \mathbf{V}_{\mathrm{in}(h)}/\mathbf{S}_{\mathrm{in}(h)}. \end{aligned}$$

It follows that x' and x'' are nilpotent.

Lemma 3.1 ([N94, Lemma 10.3]). *One has*

$$(p_3 \circ p_2)^{-1}(\Lambda(\mathbf{v}, \mathbf{w})^{st}) \subset p_1^{-1}(\Lambda(\mathbf{v}'', \mathbf{0}) \times \Lambda(\mathbf{v}', \mathbf{w})^{st}).$$

Thus, we can restrict (3.1) to Λ^{st}, forget the $\Lambda(\mathbf{v}'', \mathbf{0})$-factor and consider the quotient by $G_{\mathbf{V}}$, $G_{\mathbf{V}'}$. This yields the diagram

$$\mathcal{L}(\mathbf{v}', \mathbf{w}) \overset{\pi_1}{\leftarrow} \mathcal{F}(\mathbf{v}, \mathbf{w}; \mathbf{v} - \mathbf{v}') \overset{\pi_2}{\rightarrow} \mathcal{L}(\mathbf{v}, \mathbf{w}), \tag{3.2}$$

where

$$\mathcal{F}(\mathbf{v}, \mathbf{w}, \mathbf{v} - \mathbf{v}') \overset{\mathrm{def}}{=} \{(x, j, \mathbf{S}) \in \mathbf{F}(\mathbf{v}, \mathbf{w}; \mathbf{v} - \mathbf{v}') \mid (x, j) \in \Lambda(\mathbf{v}, \mathbf{w})^{\mathrm{st}}\}/G_{\mathbf{V}}.$$

Let $M(\mathcal{L}(\mathbf{v}, \mathbf{w}))$ be the vector space of all constructible functions on $\mathcal{L}(\mathbf{v}, \mathbf{w})$. For a subvariety Y of a variety A, let $\mathbf{1}_Y$ denote the function on A which takes the value 1 on Y and 0 elsewhere. Let $\chi(Y)$ denote the Euler characteristic of the algebraic variety Y. Then for a map π between algebraic varieties A and B, let $\pi_!$ denote the map between the abelian groups of constructible functions on A and B given by

$$\pi_!(\mathbf{1}_Y)(y) = \chi(\pi^{-1}(y) \cap Y), \ Y \subset A$$

and let π^* be the pullback map from functions on B to functions on A acting as $\pi^* f(y) = f(\pi(y))$. Then define

$$\begin{aligned} &H_i : M(\mathcal{L}(\mathbf{v}, \mathbf{w})) \to M(\mathcal{L}(\mathbf{v}, \mathbf{w})); \quad H_i f = u_i f, \\ &E_i : M(\mathcal{L}(\mathbf{v}, \mathbf{w})) \to M(\mathcal{L}(\mathbf{v} - \mathbf{e}^i, \mathbf{w})); \quad E_i f = (\pi_1)_!(\pi_2^* f), \\ &F_i : M(\mathcal{L}(\mathbf{v} - \mathbf{e}^i, \mathbf{w})) \to M(\mathcal{L}(\mathbf{v}, \mathbf{w})); \quad F_i g = (\pi_2)_!(\pi_1^* g). \end{aligned}$$

Here

$$\mathbf{u} = {}^t(u_0, \ldots, u_n) = \mathbf{w} - C\mathbf{v}$$

where C is the Cartan matrix of $\mathfrak{g}$ and we are using diagram (3.2) with $\mathbf{v}' = \mathbf{v} - \mathbf{e}^i$ where $\mathbf{e}^i$ is the vector whose components are given by $\mathbf{e}^i_j = \delta_{ij}$.

Now let φ be the constant function on $\mathcal{L}(\mathbf{0}, \mathbf{w})$ with value 1. Let $L(\mathbf{w})$ be the vector space of functions generated by acting on φ with all possible combinations of the operators F_i. Then let $L(\mathbf{v}, \mathbf{w}) = M(\mathcal{L}(\mathbf{v}, \mathbf{w})) \cap L(\mathbf{w})$.

Proposition 3.2 ([N94, Thm 10.14]). *The operators E_i, F_i, H_i on $L(\mathbf{w})$ provide the structure of the irreducible highest weight integrable representation of $\mathfrak{g}$ with highest weight $\mathbf{w}$. Each summand of the decomposition $L(\mathbf{w}) = \bigoplus_{\mathbf{v}} L(\mathbf{v}, \mathbf{w})$ is a weight space with weight $\mathbf{w} - C\mathbf{v}$. That is, with weight*

$$\sum_{i \in I} (\mathbf{w} - C\mathbf{v})_i \Lambda_i$$

where the Λ_i are the fundamental weights of $\mathfrak{g}$ (i.e. $\Lambda_i(\alpha_j) = \delta_{ij}$ where α_j is the simple root corresponding to the vertex j).

Let $X \in \operatorname{Irr} \mathcal{L}(\mathbf{v}, \mathbf{w})$ and define a linear map $T_X : L(\mathbf{v}, \mathbf{w}) \to \mathbb{C}$ as in [L92, 3.8]. The map T_X associates to a constructible function $f \in L(\mathbf{v}, \mathbf{w})$ the (constant) value of f on a suitable open dense subset of X. The fact that $L(\mathbf{v}, \mathbf{w})$ is finite-dimensional allows us to take such an open set on which *any* $f \in L(\mathbf{v}, \mathbf{w})$ is constant. So we have a linear map

$$\Phi : L(\mathbf{v}, \mathbf{w}) \to \mathbb{C}^{\operatorname{Irr} \mathcal{L}(\mathbf{v}, \mathbf{w})}.$$

The following proposition is proved in [L92, 4.16] (slightly generalized in [N94, Proposition 10.15]).

Proposition 3.3. *The map Φ is an isomorphism; for any $X \in \operatorname{Irr} \mathcal{L}(\mathbf{v}, \mathbf{w})$, there is a unique function $g_X \in L(\mathbf{v}, \mathbf{w})$ such that for some open dense subset O of X we have $g_X|_O = 1$ and for some closed $G_{\mathbf{V}}$-invariant subset $K \subset \mathcal{L}(\mathbf{v}, \mathbf{w})$ of dimension $< \dim \mathcal{L}(\mathbf{v}, \mathbf{w})$ we have $g_X = 0$ outside $X \cup K$. The functions g_X for $X \in \operatorname{Irr} \Lambda(\mathbf{v}, \mathbf{w})$ form a basis of $L(\mathbf{v}, \mathbf{w})$.*

4 Geometric Realization of the Spin Representations

We now seek to describe the irreducible components of Nakajima's quiver variety corresponding to the spin representations of the Lie algebra $\mathfrak{g}$ of type D_n. By the comment made in Section 2, it suffices to determine which irreducible components of $\Lambda(\mathbf{v}, \mathbf{w})$ are not killed by the stability condition. By Definition 2.1 and Lemma 2.4, these are precisely those irreducible components which contain points x such that

$$\dim \left(\bigcap_{h : \operatorname{out}(h) = i} \ker x_h \right) \leq \mathbf{w}_i \ \forall \ i. \tag{4.1}$$

We will consider the spin representations of highest weights Λ_{n-1} and Λ_n which correspond to $\mathbf{w} = \mathbf{w}^{n-1}$ and $\mathbf{w}^n$ respectively, where $\mathbf{w}^i_j = \delta_{ij}$.

Let $\mathcal{Y}$ be the set of all Young diagrams with strictly decreasing row lengths of length at most $n-1$, that is, the set of all strictly decreasing sequences $(l_1, \dots, l_s)$ of non-negative integers ($l_j = 0$ for $j > s$) such that $l_1 \leq n-1$. We will use the

terms Young diagram and partition interchangeably. For $Y = (l_1, \dots, l_s) \in \mathcal{Y}$ and $1 \le i \le s$, let

$$A^+_{l_i} = \begin{cases} (n - l_i, n) & \text{if } i \text{ is odd and } l_i > 1 \\ (n, n) & \text{if } i \text{ is odd and } l_i = 1 \\ (n - l_i, n - 1) & \text{if } i \text{ is even} \end{cases},$$

$$A^-_{l_i} = \begin{cases} (n - l_i, n) & \text{if } i \text{ is even and } l_i > 1 \\ (n, n) & \text{if } i \text{ is even and } l_i = 1 \\ (n - l_i, n - 1) & \text{if } i \text{ is odd} \end{cases}$$

and let

$$A^\pm_Y = \bigcup_{i=1}^{s} A^\pm_{l_i}.$$

Theorem 4.1. *The irreducible components of $\mathcal{L}(\mathbf{v}, \mathbf{w}^n)$ (resp. $\mathcal{L}(\mathbf{v}, \mathbf{w}^{n-1})$) are precisely those X_f where $f \in \tilde{Z}_{\mathbf{V}}$ such that*

$$\{(k', k) \mid f(k', k) = 1\} = A^\pm_Y$$

for some $Y \in \mathcal{Y}$, $f(k', k) = 0$ for $(k', k) \notin A^\pm_Y$, and $f((k', k)^\sim) = 0$ for all (k', k). Denote the component corresponding to such an f by $X^\pm_Y$. Thus, $Y \leftrightarrow X^\pm_Y$ is a natural 1-1 correspondence between the set $\mathcal{Y}$ and the irreducible components of $\cup_{\mathbf{v}} \mathcal{L}(\mathbf{v}, \mathbf{w}^n)$ (resp. $\cup_{\mathbf{v}} \mathcal{L}(\mathbf{v}, \mathbf{w}^{n-1})$). Here the plus signs correspond to the case $\cup_{\mathbf{v}} \mathcal{L}(\mathbf{v}, \mathbf{w}^n)$ and the minus signs correspond to the case $\cup_{\mathbf{v}} \mathcal{L}(\mathbf{v}, \mathbf{w}^{n-1})$.

Proof. We prove only the $\cup_{\mathbf{v}} \mathcal{L}(\mathbf{v}, \mathbf{w}^n)$ case. The other case in analogous. Consider the two representations $(\mathbf{V}(k'_1, k_1), x(k'_1, k_1))$ and $(\mathbf{V}(k'_2, k_2), x(k'_2, k_2))$, $1 \le k' \le k \le n - 1$ of our oriented graph as described in Section 1 where the basis of $\mathbf{V}(k'_i, k_i)$ is $\{e^i_r \mid k'_i \le r \le k_i\}$. Let W be the conormal bundle to the $G_{\mathbf{V}}$-orbit through the point

$$x_\Omega = (x_h)_{h \in \Omega} = x(k'_1, k_1) \oplus x(k'_2, k_2) \in \mathbf{E}_{\mathbf{V}(k'_1, k_1) \oplus \mathbf{V}(k'_2, k_2), \Omega}.$$

By the proof of Theorem 5.1 of [FS03], we see that for

$$x = (x_\Omega, x_{\bar\Omega}) = ((x_h)_{h \in \Omega}, (x_h)_{h \in \bar\Omega}) = (x_h)_{h \in H} \in W,$$

all of $V(k'_2, k_2)$ must be in the kernel of $x_{\bar\Omega}$ unless $k'_2 > k'_1$ and $k_2 > k_1$. If these conditions are satisfied, there exists a point $(x_\Omega, x_{\bar\Omega}) \in W$ such that $x_{h_{i,i-1}}(e^2_i) = c e^1_{i-1}$ for all $k_1 + 1 \le i \le k'_2$ (for a fixed non-zero c). We picture such a representation as in Figure 7.

Similarly, for the two represenations $(\mathbf{V}(k'_1, n), x(k'_1, n))$ and $(\mathbf{V}(k'_2, n), x(k'_2, n))$, Both $\mathbf{V}(k'_1, n)$ and $\mathbf{V}(k'_2, n)$ are in the kernel of any component x_h, $h \in \bar\Omega$ of a point in the conormal bundle to the orbit through $x(k'_1, n) \oplus x(k'_2, n)$ (since the left endpoints of the strings are the same).

However, for $k'_1 < k'_2$, there are points x in the conormal bundle to the orbit through $x(k'_1, n) \oplus x(k'_2, n-1)$ such that $x_{h_{i,i-1}}(e^2_i) = c e^1_{i-1}$ for all $k'_2 \le i \le n-1$

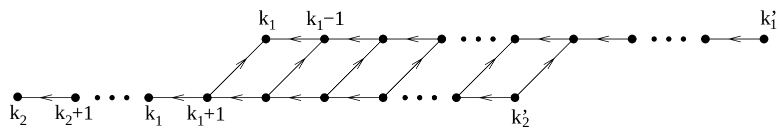

Figure 7: If $x_{h_{i,i-1}}(e_i^2) \neq 0$ for some i, the commutativity of the above diagram forces $k_2' > k_1'$ and $k_2 > k_1$. The vertices in the upper (resp. lower) string represent the basis vectors defining the representation $(\mathbf{V}(k_1', k_1), x(k_1', k_1))$ (resp. $(\mathbf{V}(k_2', k_2), x(k_2', k_2))$). A vertex labelled i represents $e_i^j \in \mathbf{V}_i$ ($j = 1, 2$). The arrows indicate the action of the obvious component of x.

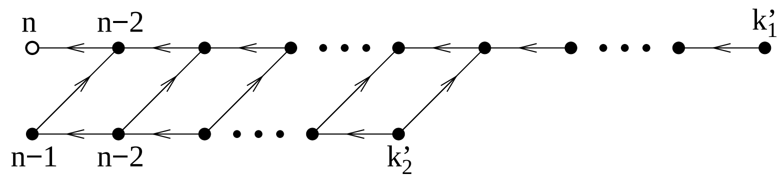

Figure 8: If $x_{h_{i,i-1}}(e_i^2) \neq 0$ for some i, the commutativity of the above diagram forces $k_2' > k_1'$. The vertices in the upper (resp. lower) string represent the basis vectors defining the representation $(\mathbf{V}(k_1', n), x(k_1', n))$ (resp. $(\mathbf{V}(k_2', n-1), x(k_2', n-1))$). A vertex labelled i represents $e_i^j \in \mathbf{V}_i$ ($j = 1, 2$). The arrows indicate the action of the obvious component of x.

and some non-zero c. These maps do not violate the moment map condition since the left endpoints of the two strings are different. Such a representation is pictured as in Figure 8. By symmetry, reversing the roles of $n-1$ and n, there are points x in the conormal bundle to the orbit through $x(k_1', n-1) \oplus x(k_2', n)$ such that $x_{h_{n,n-2}}(e_n^2) = ce_{n-2}^1$ and $x_{h_{i,i-1}}(e_i^2) = ce_{i-1}^1$ for all $k_2' \leq i \leq n-2$ and some non-zero c if and only if $k_1' < k_2'$

Now consider the representations $(\mathbf{V}(k_1', n-1), x(k_1', n-1))$ and $(\mathbf{V}(k_2', n+1), x(k_2', n+1))$. Suppose there exists a point $(x_\Omega, x_{\bar{\Omega}})$ in the conormal bundle to the orbit through the point $x(k_1', n-1) \oplus x(k_2', n+1)$ such that $(\mathbf{V}(k_2', n+1), x(k_2', n+1))$ is not contained in the kernel of $x_{\bar{\Omega}}$. Then there is some basis element $e_i^2 \in \mathbf{V}(k_2', n+1)$ that is not killed by $x_{\bar{\Omega}}$. This implies that $x_{h_{i,i-1}}(e_i^2) = ce_{i-1}^1$ for some $c \neq 0$ since $x_{h_{i,i-1}}(e_i^2)$ can have no e_{i-1}^2 component by nilpotency. As the cases above, this implies $k_2' > k_1'$.

Now, let $x = (x_\Omega, x_{\bar{\Omega}})$ lie in the conormal bundle to the point

$$\bigoplus_{i=1}^{s} x(k_i', k_i) \in \mathbf{E}_{\oplus_{i=1}^{s} \mathbf{V}(k_i', k_i), \Omega}. \tag{4.2}$$

We can assume (by reordering the indices if necessary) that $k_1' \leq k_2' \leq \cdots \leq k_s'$. By the above arguments, if $k_1 = n+1$ then both e_{n-1}^1 and e_n^1 lie in the kernel of $x_{\bar{\Omega}}$ (and hence x). But this violates the stability condition. So $k_1 \leq n$. Then we

know from the above that $e^1_{k_1}$ is in the kernel of $x_{\bar{\Omega}}$ (and hence x) since $k'_j \geq k'_1$ for all j. Thus

$$e^1_{k_1} \in \bigcap_{h:\mathrm{out}(h)=k_1} \ker x_h.$$

By the stability condition, we must then have $k_1 = n$ and there can be no other e^j_i in $\bigcap_{h:\mathrm{out}(h)=i} \ker x_h$ for any i. Suppose $k_2 = n+1$. Then we must have $x_{h_{n,n-2}}(e^2_n) = ce^1_{n-2}$. But then $x_{h_{n-2,n}} x_{h_{n,n-2}}(e^2_n) = ce^1_n \neq 0$ which violates the moment map condition at the nth vertex. So $k_2 \leq n$. Then, by the above considerations, $e^2_{k_2}$ is in $\bigcap_{h:\mathrm{out}(h)=k_2} \ker x_h$ unless $k_2 = n-1$ and $x_{h_{n-1,n-2}}(e^2_{n-1})$ is a non-zero multiple of e^1_{n-2}. Continuing in this manner, we see that we must have

$$k_i = \begin{cases} n & \text{if } i \text{ is odd} \\ n-1 & \text{if } i \text{ is even} \end{cases}$$

and $x_{h_{k_{i+1},n-2}}(e^{i+1}_{k_{i+1}})$ has non-zero e^i_{n-2} component for $1 \leq i \leq s-1$. Then by the above we must have $k'_{i+1} > k'_i$ for $1 \leq i \leq s-1$. Setting

$$l_i = \begin{cases} n - k'_i & \text{if } k'_i \leq n-2 \\ 1 & \text{if } k'_i = n-1, n \end{cases}$$

we have an irreducible component of the type mentioned in the theorem. It remains to consider the case where a summand $\tilde{x}(k', k)$ occurs with some non-zero multiplicity. Using the moment map condition and the same sort of arguments as above, one can see that x_h for $h \in \bar{\Omega}$ acting on vectors of such summands can only have non-zero components of vectors in other summands of this type or summands $x(k', k)$ for $k \leq n-2$. Thus, it follows by the nilpotency and stability conditions that no such summands can occur. The theorem follows. $\square$

The Young diagrams enumerating the irreducible components of $\cup_{\mathbf{v}} \mathcal{L}(\mathbf{v}, \mathbf{w}^n)$ can be visualized as in Figure 9. For the case of $\cup_{\mathbf{v}} \mathcal{L}(\mathbf{v}, \mathbf{w}^{n-1})$, simply interchange the labellings $n-1$ and n. Note that the vertices in our diagram correspond to the boxes in the classical Young diagram, and our arrows intersect the classical diagram edges (cf. [FS03]).

It is relatively easy to compute the geometric action of the generators E_k and F_k of $\mathfrak{g}$. We first note that for every $\mathbf{v}$, $\mathcal{L}(\mathbf{v}, \mathbf{w}^n)$ and $\mathcal{L}(\mathbf{v}, \mathbf{w}^{n-1})$ are either empty or a point. This can be seen directly or from the dimension formula for quiver varieties [N98, Cor. 3.12]. It follows that each X^+_Y (resp. X^-_Y) is equal to $\mathcal{L}(\mathbf{v}, \mathbf{w}^n)$ (resp. $\mathcal{L}(\mathbf{v}, \mathbf{w}^{n-1})$) for some unique $\mathbf{v}$ which we will denote $\mathbf{v}^+_Y$ (resp. $\mathbf{v}^-_Y$).

Lemma 4.2. *The function $g_{X^\pm_Y}$ corresponding to the irreducible component $X^\pm_Y$ where $Y \in \mathcal{Y}$ is simply $\mathbf{1}_{X^\pm_Y}$, the function on $X^\pm_Y$ with constant value one.*

Proof. This is obvious since X_Y is a point. $\square$

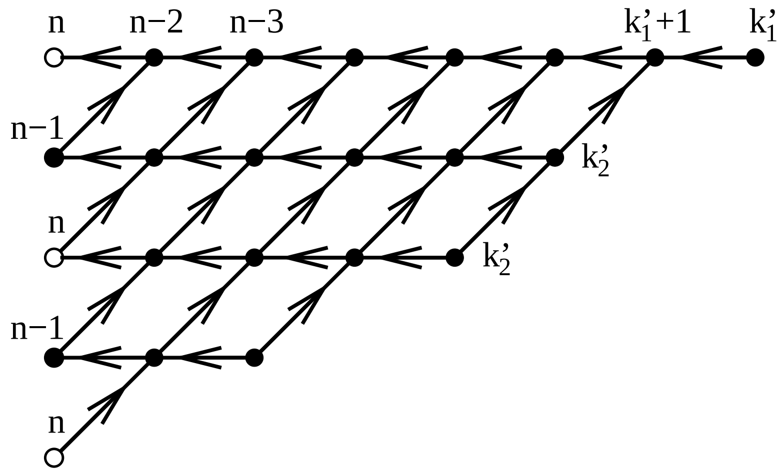

Figure 9: The irreducible components of $\cup_v \mathcal{L}(\mathbf{v}, \mathbf{w}^n)$ are enumerated by Young diagrams.

Proposition 4.3. *One has* $F_k \mathbf{1}_{X_Y^\pm} = \mathbf{1}_{X_{Y'}^\pm}$ *where* $\mathbf{v}_{Y'}^\pm = \mathbf{v}_Y^\pm + \mathbf{e}^k$ *if such a* Y' *exists and* $F_k \mathbf{1}_{X_Y^\pm} = 0$ *otherwise. Also,* $E_k \mathbf{1}_{X_Y^\pm} = \mathbf{1}_{X_{Y''}^\pm}$ *where* $\mathbf{v}_{Y''}^\pm = \mathbf{v}_Y^\pm - \mathbf{e}^k$ *if such a* Y'' *exists and* $E_k \mathbf{1}_{X_Y^\pm} = 0$ *otherwise.*

Proof. We prove the $\mathcal{L}(\mathbf{v}, \mathbf{w}^n)$ case. The case of $\mathcal{L}(\mathbf{v}, \mathbf{w}^{n-1})$ is analogous. It is clear from the definitions that $F_k \mathbf{1}_{X_Y^+} = c_1 \mathbf{1}_{X_{Y'}^+}$ and $E_k \mathbf{1}_{X_Y^+} = c_2 \mathbf{1}_{X_{Y''}^+}$ for some constants c_1 and c_2 if Y' and Y'' exist as described above and that these actions are zero otherwise. We simply have to compute the constants c_1 and c_2. Now,

$$\begin{aligned} F_k \mathbf{1}_{X_Y^+}(x) &= (\pi_2)_! \pi_1^* \mathbf{1}_{X_Y^+}(x) \\ &= \chi(\{S \mid S \text{ is } x\text{-stable, } x|_S \in X_Y^+\}) \\ &= \chi(\mathrm{pt}) \\ &= 1 \end{aligned}$$

if $x \in X_{Y'}$ where $\mathbf{v}_{Y'}^+ = \mathbf{v}_Y^+ + \mathbf{e}^k$ and zero otherwise. The fact that the above set is simply a point follows from the fact that S_k must be the sum of the images of x_h such that $\mathrm{in}(h) = k$. That is, it must be the span of all the vectors corresponding to the vertices in the column associated to S_k except the bottommost vertex. Thus $c_1 = 1$ as desired.

A similar argument shows that $c_2 = 1$. For a Young diagram Y that contains a removable vertex k (that is, a Y'' exists as described above), there is only one way to extend the unique representation (up to isomorphism) corresponding to Y'' (recall that $\mathcal{L}(\mathbf{v}_{Y''}, \mathbf{w})$ is a point) to a representation corresponding to Y – it must be the unique such representation (up to isomorphism). $\square$

We now compute the weights of the functions corresponding to the various irreducible components of the quiver variety. Let $V = \mathbb{C}^{2n}$ with basis $\{a_1, \dots, a_n, b_1, \dots, b_n\}$ and let

$$Q : V \times V \to \mathbb{C}$$

be the nondegenerate, symmetric bilinear form on V given by

$$\begin{gathered} Q(a_i, b_i) = Q(b_i, a_i) = 1 \\ Q(a_i, a_j) = Q(b_i, b_j) = 0 \\ Q(a_i, b_j) = 0 \text{ if } i \neq j. \end{gathered}$$

Then the Lie algebra $\mathfrak{g} = \mathfrak{so}_{2n}\mathbb{C}$ of type D_n consists of the endomorphisms $A : V \to V$ satisfying

$$Q(Av, w) = Q(v, Aw) = 0 \ \forall \ v, w \in V.$$

In the above basis, the Cartan subalgebra $\mathfrak{h}$ is spanned by the matrices

$$D_i = e_{i,i} - e_{n+i,n+i}$$

where $e_{i,j}$ is the matrix with a one in entry (i, j) and zeroes everywhere else. Thus the dual space $\mathfrak{h}^*$ is spanned by the functions $\{\varepsilon_i\}_{i=1}^n$ given by

$$\varepsilon_i(D_j) = \delta_{ij}.$$

The simples roots are given in this basis by

$$\begin{aligned} &\alpha_i = \varepsilon_i - \varepsilon_{i+1}, \ 1 \leq i \leq n-1 \\ &\alpha_n = \varepsilon_{n-1} + \varepsilon_n \end{aligned}$$

and

$$\begin{aligned} \Lambda_{n-1} &= (\varepsilon_1 + \cdots + \varepsilon_{n-1} - \varepsilon_n)/2 \\ \Lambda_n &= (\varepsilon_1 + \cdots + \varepsilon_{n-1} + \varepsilon_n)/2. \end{aligned}$$

Let $\lceil a \rceil$ denote the least integer greater than or equal to a and let $\lfloor a \rfloor$ denote the greatest integer less than or equal to a.

Proposition 4.4. *For a Young diagram (or partition) $Y = (\lambda_1, \dots, \lambda_s) \in \mathcal{Y}$, let $\mu = (\mu_1, \dots, \mu_t)$ be the conjugate partition. Then the weight of $\mathbf{1}_{X_Y^+}$ is*

$$\begin{aligned} &\Lambda_n - \left\lceil \frac{\mu_1}{2} \right\rceil \alpha_n - \left\lfloor \frac{\mu_1}{2} \right\rfloor \alpha_{n-1} - \sum_{i=2}^{t} \mu_i \alpha_{n-i} \\ &= \frac{1}{2} \left(\sum_{i=1}^{n-1} \varepsilon_i - \sum_{i=1}^{s} \varepsilon_{n-\lambda_i} \right) + \frac{1}{2} \begin{cases} \varepsilon_n & \textit{if } l(\lambda) \textit{ is even} \\ -\varepsilon_n & \textit{if } l(\lambda) \textit{ is odd} \end{cases} \end{aligned}$$

and the weight of $\mathbf{1}_{X_Y^-}$ *is*

$$\begin{aligned}&\Lambda_{n-1} - \left\lfloor \frac{\mu_1}{2} \right\rfloor \alpha_n - \left\lceil \frac{\mu_1}{2} \right\rceil \alpha_{n-1} - \sum_{i=2}^{t} \mu_i \alpha_{n-i} \\ &= \frac{1}{2}\left(\sum_{i=1}^{n-1} \varepsilon_i - \sum_{i=1}^{s} \varepsilon_{n-\lambda_i}\right) + \frac{1}{2}\begin{cases}\varepsilon_n & \text{if } l(\lambda) \text{ is odd} \\ -\varepsilon_n & \text{if } l(\lambda) \text{ is even}\end{cases}.\end{aligned}$$

Proof. The first expression follows easily from counting the vertices in the Young diagram and the second from switching to the bases given by the ε_i. For instance, we see that the space $V(k, n-1)$ contributes a weight

$$-\sum_{i=k}^{n-1} \alpha_i = -\sum_{i=k}^{n-1} (\varepsilon_i - \varepsilon_{i+1}) = \varepsilon_n - \varepsilon_k$$

and the space $V(k, n)$, $k \leq n-2$, contributes a weight

$$-\alpha_n - \sum_{i=k}^{n-2} \alpha_i = -(\varepsilon_{n-1} + \varepsilon_n) - \sum_{i=k}^{n-2} (\varepsilon_i - \varepsilon_{i+1}) = -\varepsilon_n - \varepsilon_k.$$

□

5 Geometric Realization of the Clifford Algebra

The standard construction of the spin representations considered above is through the use of the associated Clifford algebra. In this section, we will given a geometric realization of this algebra.

5.1 Preliminaries

We first review the neccessary details concering the Clifford algebra. Proofs can be found (for example) in [FH].

Let the vector space V and the bilinear form Q be as in the previous section. Let $C = C(Q)$ be the Clifford algebra associated to Q. That is, it is the associative algebra generated by V with relations $v \cdot v = \frac{1}{2} Q(v, v) \cdot 1$ or equivalently

$$\{v, w\} = v \cdot w + w \cdot v = Q(v, w) \ \forall \ v, w \in V.$$

Since the relations are linear combinations of elements of even degree, C inherits a $\mathbb{Z}/2\mathbb{Z}$-grading $C = C^{\text{even}} \oplus C^{\text{odd}}$. Obviously, C^{even} is a subalgebra of C.

Let $W = \operatorname{Span}\{a_1, \ldots, a_n\}$, $W' = \operatorname{Span}\{b_1, \ldots, b_n\}$ and $\bigwedge W' = \bigwedge^0 W' \oplus \cdots \oplus \bigwedge^n W'$. Recall that $b_I = b_{i_1} \wedge b_{i_2} \wedge \cdots \wedge b_{i_k}$ for $I = \{i_1 < i_2 < \cdots < i_k\}$, and with $b_\emptyset = 1$ form a basis for $\bigwedge W'$.

For $w' \in W'$, let $L_{w'} \in \operatorname{End}(\bigwedge W')$ be left multiplication by w':

$$L_{w'}(\xi) = w' \wedge \xi, \ \xi \in \bigwedge W'.$$

For $\vartheta \in (W')^*$, let $D_\vartheta \in \mathrm{End}(\bigwedge W')$ be the derivation of $\bigwedge W'$ given by

$$D_\vartheta(w'_1 \wedge \cdots \wedge w'_r) = \sum_{i=1}^{r}(-1)^{i-1}\vartheta(w'_i)(w'_1 \wedge \cdots \wedge \hat{w}'_i \wedge \ldots w'_r)$$

where $\hat{w}'_i$ means the factor w'_i is missing. Define a map $l : C \to \mathrm{End}(\bigwedge W')$ by

$$l(w') = L_{w'}, \quad l(w) = D_\vartheta, \quad w \in W, w' \in W'$$

where $\vartheta \in (W')^*$ is given by $\vartheta(w') = Q(w, w')$ for all $w' \in W'$.

Lemma 5.1. *The map l is an isomorphism of algebras $C \cong \mathrm{End}(\bigwedge W')$.*

It follows that

$$C^{\mathrm{even}} \cong \mathrm{End}({\textstyle\bigwedge}^{\mathrm{even}} W') \oplus \mathrm{End}({\textstyle\bigwedge}^{\mathrm{odd}} W').$$

From now on, we suppress the isomophism l and identify C with $\mathrm{End}(\bigwedge W')$. We can view C as a Lie algebra with bracket given by the commutator.

Proposition 5.2. *The map $\mathfrak{g} \to C^{even}$ given on generators by*

$$\begin{aligned}
E_k &\mapsto b_{k+1}a_k = L_{b_{k+1}} D_{b_k^*}, \ 1 \le k \le n-1 \\
F_k &\mapsto b_k a_{k+1} = L_{b_k} D_{b_{k+1}^*}, \ 1 \le k \le n-1 \\
E_n &\mapsto a_n a_{n-1} = D_{b_n^*} D_{b_{n-1}^*} \\
F_n &\mapsto b_{n-1} b_n = L_{b_{n-1}} L_{b_n}
\end{aligned}$$

is an injective morphism of Lie algebras. This gives $\bigwedge W'$ the structure of a $\mathfrak{g}$-module and

$${\textstyle\bigwedge} W' \cong L_{\Lambda_{n-1}} \oplus L_{\Lambda_n}$$

as $\mathfrak{g}$-modules where L_λ is the irreducible representation of highest weight λ. In particular

$$\begin{aligned}
{\textstyle\bigwedge}^{even} W' &\cong \begin{cases} L_{\Lambda_n} & \text{if } n \text{ is even} \\ L_{\Lambda_{n-1}} & \text{if } n \text{ is odd} \end{cases}, \\
{\textstyle\bigwedge}^{odd} W' &\cong \begin{cases} L_{\Lambda_n} & \text{if } n \text{ is odd} \\ L_{\Lambda_{n-1}} & \text{if } n \text{ is even} \end{cases}.
\end{aligned}$$

The weight of the natural basis vector b_I is

$$\frac{1}{2}\left(\sum_{i \notin I} \varepsilon_i - \sum_{j \in I} \varepsilon_j\right).$$

5.2 Geometric Realization

We now present a geometric construction of the Clifford algebra. Recall the maps

$$\mathcal{L}(\mathbf{v}',\mathbf{w}) \overset{\pi_1}{\leftarrow} \mathcal{F}(\mathbf{v},\mathbf{w};\mathbf{v}-\mathbf{v}') \overset{\pi_2}{\rightarrow} \mathcal{L}(\mathbf{v},\mathbf{w})$$

of Diagram 3.2. For $1 \le k \le n-1$, define elements $\mathbf{y}^k, \mathbf{z}^k \in \mathbb{Z}^n_{\ge 0}$ by

$$\mathbf{y}^k_i = \begin{cases} 1 & \text{if } k \le i \le n-1 \\ 0 & \text{otherwise} \end{cases}$$
$$\mathbf{z}^k_i = \begin{cases} 1 & \text{if } k \le i \le n-2 \text{ or } k = n \\ 0 & \text{otherwise} \end{cases}.$$

Then consider the maps

$$\bigcup_v \mathcal{L}(\mathbf{v},\mathbf{w}) \overset{\pi_1}{\leftarrow} \bigcup_v \mathcal{F}(\mathbf{v},\mathbf{w};\mathbf{y}^k) \cup \bigcup_v \mathcal{F}(\mathbf{v},\mathbf{w};\mathbf{z}^k) \overset{\pi_2}{\rightarrow} \bigcup_v \mathcal{L}(\mathbf{v},\mathbf{w}) \tag{5.1}$$

where π_1 and π_2 are as above. Now, for $\mathbf{v} \in \mathbb{Z}^n_{\ge 0}$, define $\mathbf{v}^*$ by

$$\mathbf{v}^*_i = \mathbf{v}_i,\ 1 \le i \le n-2$$
$$\mathbf{v}^*_{n-1} = \mathbf{v}_n, \quad \mathbf{v}^*_n = \mathbf{v}^*_{n-1}.$$

Note that $\mathbf{v} \mapsto \mathbf{v}^*$ is an involution. Let

$$\kappa : \mathcal{L}(\mathbf{v},\mathbf{w}) \to \mathcal{L}(\mathbf{v}^*,\mathbf{w}^*)$$

be the map that "switches" the vertices $n-1$ and n. Specifically, for $[(x,j)] \in \mathcal{L}(\mathbf{v},\mathbf{w})$ ($[(x,j)]$ denotes the $G_{\mathbf{V}}$-orbit through the point (x,j)), $\kappa([(x,j)]) = [(x',j')]$ where

$$\begin{aligned} x'_h &= x_h \text{ for } h \ne h_{n-1,n-2}, h_{n,n-2}, h_{n-2,n-1}, h_{n-2,n} \\ x'_{h_{n-1,n-2}} &= x_{h_{n,n-2}},\ x'_{h_{n-2,n-1}} = x_{h_{n-2,n}} \\ x'_{h_{n,n-2}} &= x_{h_{n-1,n-2}},\ x'_{h_{n-2,n}} = x_{h_{n-2,n-1}} \\ j'_i &= j_i \text{ for } i \ne n-1, n \\ j'_{n-1} &= j_n \\ j'_n &= j_{n-1}. \end{aligned}$$

Note that κ^2 is the identity map.

Recall that the set of $\mathbf{v}$ such that $\mathcal{L}(\mathbf{v},\mathbf{w})$ is non-empty is in 1-1 correspondence with the set $\mathcal{Y}$. Denote the Young diagram corresponding to such a $\mathbf{v}$ by $Y_{\mathbf{v}}$. Recall that each row in the Young diagram corresponds to a string of vertices of the Dynkin diagram of $\mathfrak{g}$. We say that the *endpoint* of the row is the lowest index of the vertices appearing in that string with one exception: we say that the endpoint of the string consisting of the single vertex n is $n-1$. For a Young diagram Y, let $l_i(Y)$ be the number of rows of Y with endpoint strictly

less than i (or, equivalently, the number rows of length greater that $n-i$) and $l(Y)$ the number of rows of Y. Note that

$$l(Y_{\mathbf{v}}) = \mathbf{v}_{n-1} + \mathbf{v}_n.$$

Now, for $\mathbf{w} = \mathbf{w}^{n-1}$ or $\mathbf{w}^n$ and for $1 \le k \le n-1$ define

$$\begin{aligned} a_k &: M(\mathcal{L}(\mathbf{v},\mathbf{w})) \to M(\mathcal{L}(\mathbf{v}-\mathbf{y}^k,\mathbf{w}^*)) \oplus M(\mathcal{L}(\mathbf{v}-\mathbf{z}^{\mathbf{k}},\mathbf{w}^*)) \\ b_k &: M(\mathcal{L}(\mathbf{v},\mathbf{w})) \to M(\mathcal{L}(\mathbf{v}+\mathbf{y}^k,\mathbf{w}^*)) \oplus M(\mathcal{L}(\mathbf{v}+\mathbf{z}^k,\mathbf{w}^*)) \end{aligned}$$

by

$$\begin{aligned} a_k(f) &= (-1)^{l_k(Y_{\mathbf{v}})}\kappa^*(\pi_1)_!\pi_2^* f \\ b_k(f) &= (-1)^{l_k(Y_{\mathbf{v}})}\kappa^*(\pi_2)_!\pi_1^* f \end{aligned}$$

where we have used Diagram 5.1. Define

$$\begin{aligned} a_n &: M(\mathcal{L}(\mathbf{v},\mathbf{w})) \to M(\mathcal{L}(\mathbf{v},\mathbf{w}^*)) \\ b_n &: M(\mathcal{L}(\mathbf{v},\mathbf{w})) \to M(\mathcal{L}(\mathbf{v},\mathbf{w}^*)) \end{aligned}$$

by

$$a_n(f) = \begin{cases} (-1)^{l(Y_{\mathbf{v}})}\kappa^* f & \text{if } \dim \mathbf{w}_n + l(Y_{\mathbf{v}}) \text{ is even} \\ 0 & \text{if } \dim \mathbf{w}_n + l(Y_{\mathbf{v}}) \text{ is odd.} \end{cases}$$

$$b_n(f) = \begin{cases} (-1)^{l(Y_{\mathbf{v}})}\kappa^* f & \text{if } \dim \mathbf{w}_n + l(Y_{\mathbf{v}}) \text{ is odd} \\ 0 & \text{if } \dim \mathbf{w}_n + l(Y_{\mathbf{v}}) \text{ is even.} \end{cases}$$

For $\mathbf{v} = \mathbf{v}_Y^{\pm}$ for some $Y \in \mathcal{Y}$, note that there exists a Y^{+k} such that $\mathbf{v}_{Y^{+k}}^{\pm} = \mathbf{v}_Y^{\pm} + \mathbf{y}^k$ or $\mathbf{v}_Y^{\pm} + \mathbf{z}^k$ if and only if Y does not contain a row with endpoint k. If this is the case, then Y^{+k} is obtained from Y by adding such a row and if $\mathbf{v}_{Y^{+k}}^{\pm} = \mathbf{v}_Y^{\pm} + \mathbf{y}^k$, then $\mathcal{L}(\mathbf{v}_Y^{\pm} + \mathbf{z}^k)$ is empty and vice versa. Similarly, there exists a Y^{-k} such that $\mathbf{v}_{Y^{-k}}^{\pm} = \mathbf{v}_Y^{\pm} - \mathbf{y}^k$ or $\mathbf{v}_Y^{\pm} - \mathbf{z}^k$ if and only if Y contains a row with endpoint k. If this is the case, then Y^{-k} is obtained from Y by removing such a row and if $\mathbf{v}_{Y^{-k}}^{\pm} = \mathbf{v}_Y^{\pm} - \mathbf{y}^k$, then $\mathcal{L}(\mathbf{v}_Y^{\pm} - \mathbf{z}^k)$ is empty and vice versa.

Proposition 5.3. *Using Diagram 5.1,*

$$(\pi_1)_!\pi_2^*\mathbf{1}_{X_Y^{\pm}} = \begin{cases} \mathbf{1}_{X_{Y^{-k}}^{\pm}} & \text{if } Y \text{ contains a row with endpoint } k \\ 0 & \text{otherwise} \end{cases},$$

$$(\pi_2)_!\pi_1^*\mathbf{1}_{X_Y^{\pm}} = \begin{cases} \mathbf{1}_{X_{Y^{+k}}^{\pm}} & \text{if } Y \text{ does not contain a row with endpoint } k \\ 0 & \text{otherwise} \end{cases},$$

$$\kappa^*\mathbf{1}_{X_Y^{\pm}} = \mathbf{1}_{X_Y^{\mp}}.$$

Proof. It is clear from the definitions that $(\pi_1)_!\pi_2^*\mathbf{1}_{X_Y^\pm} = c_{-k}\mathbf{1}_{X_{Y-k}^\pm}$ for some constant c_{-k} if Y contains a row with endpoint k and is zero otherwise. Similarly, it is clear that $(\pi_2)_!\pi_1^*\mathbf{1}_{X_Y^\pm} = c_{+k}\mathbf{1}_{X_{Y+k}^\pm}$ for some constant c_{+k} if Y does not contain a row with endpoint k and is zero otherwise. It remains to compute the constants c_{-k} and c_{+k}. Assume Y does not contain a row with endpoint k. Then,

$$\begin{aligned}(\pi_2)_!\pi_1^*\mathbf{1}_{X_Y^\pm}(x) &= \chi(\{S \mid S \text{ is } x\text{-stable, } x|_S \in X_Y^\pm\}) \\ &= \chi(\mathrm{pt}) \\ &= 1\end{aligned}$$

if $x \in X_{Y+k}^\pm$ and is zero otherwise. The fact that the above set is a point follows from the fact that x-stability implies that S must be the span of the vectors corresponding to the vertices of the Young diagram obtained from Y^{+k} by removing the lowest vertex of each of the first $n-k$ columns. This is because the vector corresponding to a given vertex lies in the smallest subspace of $\mathbf{V}$ containing any vector below it, in the same column. Thus $c_{+k} = 1$.

A similar argument shows that $c_{-k} = 1$. For a Young diagram Y that contains a row with endpoint k, there is only one way to extend the unique representation (up to isomorphism) corresponding to Y^{-k} (recall that $\mathcal{L}(\mathbf{v}_Y, \mathbf{w})$ is a point) to a representation corresponding to Y – it must be the unique such representation (up to isomorphism). $\square$

Theorem 5.4. *The operators a_k, b_k on $L(\mathbf{w}^{n-1}) \oplus L(\mathbf{w}^n)$ provide the structure of a representation of the Clifford algebra C. Moreover, as representations of C,*

$$L(\mathbf{w}^{n-1}) \oplus L(\mathbf{w}^n) \cong \bigwedge W. \tag{5.2}$$

Proof. For $Y = (n-1 \geq l_1 > \cdots > l_s \geq 1) \in \mathcal{Y}$, let

$$I_Y^+ = \begin{cases} \{l_s, \dots, l_1\} & \text{if } s \text{ is even} \\ \{l_s, \dots, l_1, n\} & \text{if } s \text{ is odd} \end{cases}$$

$$I_Y^- = \begin{cases} \{l_s, \dots, l_1\} & \text{if } s \text{ is odd} \\ \{l_s, \dots, l_1, n\} & \text{if } s \text{ is even} \end{cases}.$$

Both sides of (5.2) have dimension 2^n. We identify the two via the map $\mathbf{1}_{X_Y^\pm} \mapsto b_{I_Y^\pm}$. For a set I of indices between 1 and n that does not contain the index k, let I^{+k} be the set obtained by adding k. For a set I of indices between 1 and n that contains the index k, let I^{-k} be the set obtained by removing k. We view such sets of indices as a partition or Young diagram as follows: for $I = \{i_1, \dots, i_s\}$, let the corresponding Young diagram have rows of length $n - i_j$ (with endpoint i_j) for those $1 \leq j \leq s$ such that $i_j < n$. Thus $l_i(I)$ is defined – it is the number

of indices of I strictly less than i. Now, for $1 \le k \le n$,

$$a_k(b_{I_Y^\pm}) = D_{b_k^*}(b_{I_Y^\pm}) = \begin{cases} (-1)^{l_k(I_Y)} b_{(I_Y^\pm)^{-k}} & \text{if } I_Y^\pm \text{ contains } k \\ 0 & \text{if } I_Y^\pm \text{ does not contain } k \end{cases},$$

$$b_k(b_{I_Y^\pm}) = L_{b_k}(b_{I_Y^\pm}) = \begin{cases} (-1)^{l_k(I_Y)} b_{(I_Y^\pm)^{+k}} & \text{if } I_Y^\pm \text{ does not contain } k \\ 0 & \text{if } I_Y^\pm \text{ contains } k \end{cases}.$$

Note that $(I_Y^\pm)^{\pm k} = I_{Y^{\pm k}}^\mp$ for $1 \le k \le n-1$, $(I_Y^\pm)^{+n} = I_Y^\mp$ if $I_Y^\pm$ does not contain n and $(I_Y^\pm)^{-n} = I_Y^\mp$ if $I_Y^\pm$ contains n.

Now, recalling that a plus sign indicates the case $\mathbf{w} = \mathbf{w}^n$ and a minus sign indicates the case $\mathbf{w} = \mathbf{w}^{n-1}$, we see that $I_Y^\pm$ contains an n if and only if $\mathbf{w}_n + l(Y) = \mathbf{w}_n + l_n(Y)$ is even. Thus the result follows from the definition of the geometric action of the a_k and b_k and Proposition 5.3. □

We now show that the geometric action of the Clifford algebra we have constructed is a natural extension of the geometric action of $\mathfrak{g}$.

Proposition 5.5. *As operators on $L(\mathbf{w}^{n-1}) \oplus L(\mathbf{w}^n)$,*

$$\begin{aligned} E_k &= b_{k+1}a_k, \quad 1 \le k \le n-1 \\ F_k &= b_k a_{k+1}, \quad 1 \le k \le n-1 \\ E_n &= a_n a_{n-1} \\ F_n &= b_{n-1} b_n. \end{aligned}$$

Proof. It suffices to show that the relations hold on the $\mathbf{1}_{X_Y^\pm}$ since these functions span $L(\mathbf{w}^{n-1}) \oplus L(\mathbf{w}^n)$. Note that we can remove a vertex with degree k from Y (and be left with an element of $\mathcal{Y}$) if and only if it has a row with endpoint k but no row with endpoint $k+1$. Let Y be such a diagram. For $1 \le k \le n-1$,

$$\begin{aligned} b_{k+1}a_k \mathbf{1}_{X_Y^\pm} &= (-1)^{l_k(Y)} b_{k+1} \kappa^*(\pi_1)_! \pi_2^* \mathbf{1}_{X_Y^\pm} \\ &= (-1)^{l_k(Y)} b_{k+1} \kappa^* \mathbf{1}_{X^\pm_{Y^{-k}}} \\ &= (-1)^{l_k(Y)} b_{k+1} \mathbf{1}_{X^\mp_{Y^{-k}}} \\ &= (-1)^{l_k(Y)} (-1)^{l_{k+1}(Y^{-k})} \kappa^*(\pi_2)^! \pi_1^* \mathbf{1}_{X^\mp_{Y^{-k}}} \\ &= (-1)^{l_k(Y)} (-1)^{l_{k+1}(Y^{-k})} \kappa^* \mathbf{1}_{X^\mp_{(Y^{-k})^{+(k+1)}}} \\ &= \mathbf{1}_{X^\pm_{(Y^{-k})^{+(k+1)}}} \end{aligned}$$

But $(Y^{-k})^{+(k+1)}$ is precisely the Young diagram obtained from Y by removing a vertex k and so $b_{k+1}a_k \mathbf{1}_{X_Y^\pm} = E_k \mathbf{1}_{X_Y^\pm}$. The proof of the other three equations is analogous. □

6 Extensions

The results of this paper can be easily extended to type D_∞. One obtains an enumeration of the irreducible components of Nakajima's quiver variety by Young diagrams of stricly decreasing row lengths but with no condition on maximum row length. The geometric Lie algebra and Clifford algebra actions can be easily computed and are analogous to those obtained for type D_n.

If one is interested in the q-deformed analogues of the Clifford algebra and spin representations (cf. [DF94]), this can also be handled by the methods of the current paper. One simply has to consider the quiver varieties over a finite field instead of over $\mathbb{C}$.

To treat arbitrary finite dimensional irreducible representations instead of just the spin representations considered here, somewhat different techniques are required. This situation will be examined in a upcoming paper.

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